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Uniform Approximation in the Semiring of Positive Continuous Functions

KAREL PRIKRY[1] AND IAN RICHARDS

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

We consider a problem proposed by Jurkat and Lorentz [1]. Let $C^+(X)$ be the set of all nonnegative real valued continuous functions on a compact Hausdorff space X. Take a *semiring* R of functions in $C^+(X)$: i.e., a subset of $C^+(X)$ which is closed under addition, multiplication, and multiplication by nonnegative scalars. The problem is to determine conditions under which R is dense in $C^+(X)$ with respect to the topology of uniform convergence. Ideally, these conditions should be analogous to the "separating points" hypothesis in the Stone–Weierstrass theorem.

Jurkat and Lorentz proposed a convexity condition [1, Sect. 1.3] which is clearly necessary for the density of R. They showed that their condition is sufficient in certain special cases and raised the question of whether it is sufficient in general. We show in this paper that the answer is no. Moreover, we give a counterexample which works for the most important spaces studied in classical analysis (e.g., the interval [0, 1]). This example can also be adapted to refute a much stronger convexity hypothesis than that made in [1] (cf. condition (3) below). To summarize our conclusions: The main difference between rings and semirings is the presence in the former of a subtraction operation. Our results suggest that convexity type hypotheses can not take the place of subtraction in establishing approximation theorems.

Convexity hypotheses. The convexity condition (1.3) proposed by Jurkat and Lorentz is a natural generalization of the "separation of points" assumption in the Stone–Weierstrass theorem. It asserts that:

For any finite set of distinct points $x_0, x_1, \ldots, x_n \in X$ and any set of numbers $\alpha_k > 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, there exists a function $g \in R$ such that

$$g(x_0) > g(x_1)^{\alpha_1} \cdots g(x_n)^{\alpha_n}. \tag{1}$$

A slightly stronger condition ((1.4) in [1]) is the following:

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For any finite set of distinct points $x_0, x_1, \ldots, x_n \in X$, there exists a function $h \in R$ such that

$$h(x_0) > h(x_k) \qquad \text{for} \quad 1 \leqslant k \leqslant n. \tag{2}$$

We will give a counterexample to the sufficiency of (2) and a-fortiori to that of (1). It might be suspected that the difficulty with conditions (1) and (2) is that they involve only a finite number of points at a time. However, our examples show that even the consideration of infinite sets of points in inequalities of type (2) will not suffice. Thus we might assume a global maximum for every point $x_0 \in X$, i.e.:

For any point $x_0 \in X$, there exists a function $h \in R$ such that

$$h(x_0) > h(x) \qquad \text{for all } x \neq x_0 \text{ in } X. \tag{3}$$

This hypothesis looks at first glance very strong. For, since we can replace h by a high power of h (remembering that R is a semiring), we can construct narrow "pulses" above each point x_0. More precisely, (3) implies the condition:

For any point $x_0 \in X$, any neighborhood U of x_0, and any number $\epsilon > 0$, there exists a function $h \in R$ such that

$$\begin{aligned} & h(x_0) = 1, \quad h(x) \leqslant 1 \qquad \text{for all } x, \\ \text{and} \qquad & h(x) < \epsilon \qquad \text{for all } x \in X - U. \end{aligned} \tag{3'}$$

One might expect that such pulses could be superimposed to approximate any positive continuous function. Nevertheless this is false, as the following theorem shows.

THEOREM. *Let X be a compact metric space. Let us say that "condition* (2) *is sufficient for X" if every semiring $R \subseteq C^+(X)$ which satisfies* (2) *is dense in $C^+(X)$. Similarly define "*(3) *is sufficient for X." Then*:

(a) *Condition* (2) *is sufficient for X if and only if X is a finite set.*

(b) *Condition* (3) *is sufficient for X if and only if X is a countable set.*

Proof of Theorem, Part (a)

The sufficiency is almost trivial. Thus let X be a finite set. Condition (2) implies that for each point $x_0 \in X$, there is a function $h \in R$ with a strict maximum at x_0. There is no loss of generality in assuming that $h(x_0) = 1$, $h(x) < 1$ for $x \neq x_0$. Now replacing h be a suitably high power of h, we can have for any preassigned $\epsilon > 0$:

$$h(x_0) = 1 \qquad \text{and} \qquad h(x) < \epsilon \qquad \text{for} \quad x \neq x_0.$$

Since there are only finitely many points in X, we can use positive linear combinations of such "pulse functions" h to approximate any nonnegative real valued function. This proves the sufficiency.

For the necessity, let X be any compact metric space which contains infinitely many points. Then there must be at least one limit point $p_\infty \in X$ and (since X is a metric space) a sequence of distinct points $p_0, p_1, p_2, \ldots$ converging to p_∞ in X. In other words, X contains a closed subset $\{p_0, p_1, p_2, \ldots\} \cup \{p_\infty\}$ homeomorphic to the ordinal number $\omega + 1$. (Recall that $\omega + 1$ consists of $\{0, 1, 2, \ldots\} \cup \{\infty\}$ with ∞ as its only limit point.)

Now it is not difficult to see that it suffices to give a counterexample to (2) for the special space $\omega + 1$. For we can imbed $\omega + 1$ in X, and use the Tietze extension theorem: Suppose we have a semiring R_0 on $\omega + 1$ which gives the desired counterexample on that space. Let R be the set of all functions in $C^+(X)$ which are extensions of functions in R_0. Then R cannot be dense in $C^+(X)$ since by assumption R_0 is not dense in $C^+(\omega + 1)$. But it is easy to verify, using the Tietze theorem, that if the condition (2) is satisfied for R_0 on $\omega + 1$, it extends to R on X.

Special Construction, *Part* (a)

Let X be the ordinal number $\omega + 1$ (described above). We build a counterexample to condition (2) on X. Let R be the semiring generated by the following set G of functions (i.e., R consists of all positive linear combinations of products of functions in G).

The generating set G of functions:

1. all nonnegative functions vanishing at ∞;
2. the sequence of functions $f_0, f_1, f_2, \ldots$ defined by

$$\begin{aligned} f_n(k) &= 0 && \text{for} \quad k < n, \\ &= 1 && \text{for} \quad k = n, \\ &= 1/2^{n+1} && \text{for} \quad k > n \text{ including } k = \infty. \end{aligned}$$

We now consider the multiplicative semigroup P generated by the products (including powers) of the functions in G. (This turns out to be the crucial step; for the semiring R involves only positive *linear* combinations of functions in P.) Actually the functions in P do not look very different from those in G.

First, any product of a function in the set 1 (vanishing at ∞) with any other function gives a function vanishing at ∞.

Second, as to the functions f_n, we will see later that the constant $1/2^{n+1}$ which appears there could be replaced by any *smaller* constant without harm to our result. Taking a power of f_n produces just such a smaller constant, while leaving the values 0 and 1 fixed.

Finally, consider a product of powers of different f_n: let

$$f_a^A f_b^B \cdots f_z^Z$$

be such a product, where the indices $a < b < \cdots < z$ are arranged in increasing order. A little thought shows that this product is just

$$\text{Constant} \cdot f_z^Z$$

where the constant involves the other terms $f_a^A \cdots f_y^Y$. Thus we have shown:

Every function in the multiplicative semigroup P generated by G either vanishes at ∞ or else has the form

$$\text{Constant} \cdot f_n^Z$$

for some constant $\geqslant 0$ and some integers n and Z; moreover f_n^Z can be written:

$$\begin{aligned} f_n^Z &= 0 && \text{for } f < n, \\ &= 1 && \text{for } k = n, \\ &= 1/2^{Z(n+1)} && \text{for } k > n. \end{aligned}$$

Now we are close to our goal. We must show (i) that the functions in R satisfy condition (2) above, but (ii) R is not dense in $C^+(\omega + 1)$. For (i): the functions in the generating set G already suffice. If the point x_0 in (2) is not ∞, this is clear, since all functions vanishing at ∞ belong to G. If $x_0 = \infty$, then we take f_N where N is chosen larger than any of the values $x_1, x_2, \ldots, x_n$.

Following Jurkat and Lorentz, we use a measure theoretic argument to show that R is not dense in $C^+(\omega + 1)$. However, in this context, the measure theory involved is completely elementary. Let μ and ν be the two probability measures defined by:

$$\mu = \text{the Dirac measure at } \infty,$$

$$\nu(k) = 1/2^{k+1}, \qquad \nu(\infty) = 0.$$

Then one readily verifies, from the structure theorem for P proved above, that

$$\int f\, d\nu \geqslant \int f\, d\mu \qquad \text{for all } f \in P.$$

Since the semiring R consists of positive *linear* combinations of functions in P, the same inequality holds for all $f \in R$. Thus R cannot be dense in $C^+(\omega + 1)$; in fact any function f for which $\int f\, d\mu > \int f\, d\nu$ lies outside the closure of R. This completes the proof of part (a).

Proof of Theorem, Part (b)

Here the proof of necessity follows that in part (a), except that a Cantor set replaces the set $\omega + 1$, and the special construction on this Cantor set is a little more difficult. As before, we begin with the sufficiency. Thus suppose that the space X is countable. Now, following Jurkat and Lorentz [1, Sect. 1.1], we use the Hahn–Banach theorem to derive the criterion:

A semiring R is dense in $C^+(X)$ if and only if, for every finite signed Borel measure μ on X, the relation

$$\int_X f\, d\mu \geqslant 0 \qquad \text{for all } f \in R$$

implies that μ is a positive measure.

Here the space X is countable, and so every measure μ on X is atomic. Suppose μ is not positive. Then there is a point $x_0 \in X$ for which $\mu(x_0) < 0$. By countable additivity, there is a neighborhood U of x_0 such that $|\mu|(U - \{x_0\}) < |\mu(x_0)|/2$. Applying condition (3′) which is a corollary of (3), we see that there exists a function $f \in R$ such that $\int f\, d\mu < 0$. So by the Hahn–Banach criterion, (3) implies that R is dense in $C^+(X)$. This proves the sufficiency.

For the necessity, let X be any compact metric space which contains uncountably many points. It is well known that such a space must contain a closed subset homeomorphic to the Cantor set. As in the proof of part (a) above, we can reduce the problem to the case where X *is* a Cantor set, and then use the Tietze extension theorem. There is only one technicality which makes the extension here a little more difficult. The condition (3) above refers to a unique absolute maximum at the point x_0. However the derived condition (3′) does not involve uniqueness. The way around the "uniqueness problem" is as follows: we use condition (3′) and (temporarily) ignore (3). Then we obtain a counterexample R satisfying (3′) although not necessarily (3). But then the closure $\bar{R}$ of R in $C^+(X)$ still provides a counterexample to the approximation theorem (if R is not dense in $C^+(X)$, then neither is $\bar{R}$). And, since X is a metric space, it is easy to superimpose countably many "pulses" (using condition (3′)) so as to obtain a function $f \in \bar{R}$ with an absolute maximum at x_0. *From now on, we will pay no attention to condition* (3), *but instead use condition* (3′).

Special Construction, Part (b)

We identify the Cantor set X with the collection of infinite sequences of 0's and 1's endowed with the product topology. Let $\mathscr{B}$ be the open base consisting of sets B_s, where s ranges over finite sequences of 0's and 1's, and B_s is the set of all infinite sequences extending s. If s has length k, then we refer to B_s as a set of the kth generation; there are 2^k such sets.

We now construct the two probability measures which will play the roles of μ and ν in part (a); we will want

$$\int f\,d\nu \geqslant \int f\,d\mu \qquad \text{for all } f \in R.$$

For μ we take the standard "fair coin" or "Lebesgue" product measure on $X = \{0, 1\}^\omega$. The measure ν will be atomic: for each finite sequence s as above, we pick an arbitrary point $x_s \in B_s$ and let the measure $\nu(x_s) = 1/4^k$, where k is the length of s. [We could make a canonical choice for the x_s, using certain obvious points in the traditional "middle third" representation of the Cantor set; but there is no need for this.]

We now define inductively the generating set G for our semiring R. The set G will consist of an infinite sequence of finite sets of functions; the set of functions constructed at the nth stage will be called G_n. Actually, every function $f \in G_n$ will be the characteristic function of some clopen set $E = E(f)$. Thus we can view G_n alternatively either as a family of functions or as a family of sets. Finally, we will use a certain strictly increasing sequence of integers $k(n)$, whose inductive definition will be given below. Now we describe the (finite) family of sets $E = E(f)$ belonging to G_n.

(A) Every set $E \in G_n$ is the union of either one or two basic sets B_s of the $k(n)$th generation.

(B) The set E in (A) above is contained within a single basic set B_s of the $k(n-1)$st generation.

So far we have merely given some properties of the sets E. Now, guided by these conditions, we describe them exactly:

Fix attention on a single set B_s in the $k(n-1)$st generation. Let E' denote an *arbitrary* basis set of the $k(n)$th generation contained within B_s. Let E'' denote that *fixed* one of the sets E' which contains the point x_s. (Recall that the points $x_s \in B_s$ are those which support the measure ν.) Then we choose for our sets E all sets of the form $E = E' \cup E''$. Here E' may equal E''. We carry out this operation for all sets B_s of the $k(n-1)$st generation. The collection of sets so generated constitutes G_n.

As a consequence of our construction we have, for any sets $E \in G_n$:

(C) The intersection of two distinct sets $E_1 = E_1' \cup E''$ and $E_2 = E_2' \cup E''$ contained in the same B_s is just E''. This is also a set belonging to G_n.

(C′) The intersection of sets E coming from different B_s is empty.

Now we describe the way $k(n)$ is determined from $k(n-1)$. For each of the $2^{k(n-1)}$ sets B_s in the $k(n-1)$st generation, we look at the point x_s which supports part of the measure ν: recall that $\nu(x_s) = 1/4^{k(n-1)}$. Similarly the μ-measure of the basic sets of the kth generation is $1/2^k$. Putting in an extra factor of $\frac{1}{2}$ (for the two sets E' and E'') we define $k(n)$ to be the least integer such that

$$1/2^{k(n)} \leqslant (1/2)(1/4^{k(n-1)}).$$

[We could make this explicit, by letting $k(n) = 2^n - 1$; but there is no need for it.]

As a consequence of this we have, for any set $E = E' \cup E''$ belonging to G_n:

$$\nu(E) \geqslant \nu(E'') \geqslant \nu(x_s) = 1/4^{k(n-1)},$$

whereas

$$\mu(E) \leqslant \mu(E') + \mu(E'') = 2(1/2^{k(n)}),$$

so that by our assumptions on $k(n)$

$$\mu(E) \leqslant \nu(E) \qquad \text{for all } E \in G_n . \tag{*}$$

We now let G be the union of the G_n constructed above. We must verify that G contains enough functions to satisfy the "existence of pulses" condition (3′). Take any point $x_0 \in X$. There is some basis set B_s of the $k(n-1)$st generation which contains x_0. Within this B_s, there is some set E' of the $k(n)$th generation which contains x_0, and hence some set $E = E' \cup E''$ which contains x_0. Let h be the characteristic function of E. Then h satisfies (3′), where $\epsilon = 0$ and the neighborhood U equals B_s (which can be made as small as we please by letting $n \to \infty$). Now, having described the generating set G, we let P denote the multiplicative semigroup spanned by G. We have the striking result:

$$P = G,$$

i.e., the operation of multiplication on G produces no new functions. (We ignore the function which is identically zero.) To prove this: First, since each function $f \in G$ takes only the values 0 and 1, every power of f is equal to f itself. Second, for the products of distinct f we have:

If $f \in G_m$ and $g \in G_n$ with $m < n$, then either $fg = g$ or $fg = 0$. This follows from (A) and (B) above.

If f and g both belong to G_n, then either $fg = 0$ or fg also belongs to G_n (although it may be different from f and g). This follows from (C) and (C′) above.

We have already noted that G satisfies condition (3′), and how condition (3′) leads to (3). On the other hand, since $P = G$, and R consists of positive linear combinations of functions in P, we have from (*) above:

$$\int f\, d\nu \geqslant \int f\, d\mu \qquad \text{for all } f \in R.$$

Since ν and μ are distinct probability measures on X, this shows that R is not dense in $C^+(X)$.

References

1. W. B. Jurkat and G. G. Lorentz, Uniform Approximation by Polynomials with Positive Coefficients, *Duke J.* **28** (1961), 463–473.